The structure and evolution of stars

Lecture 5: The equations of stellar structure

Karl Schwarschild (1873-1916)

Introduction and recap

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity
2) **Conservation of mass**
3) **Conservation of energy :** at each radius, the change in the energy flux = local rate of energy release
4) **Equation of energy transport :** relation between the energy flux and the local gradient of temperature

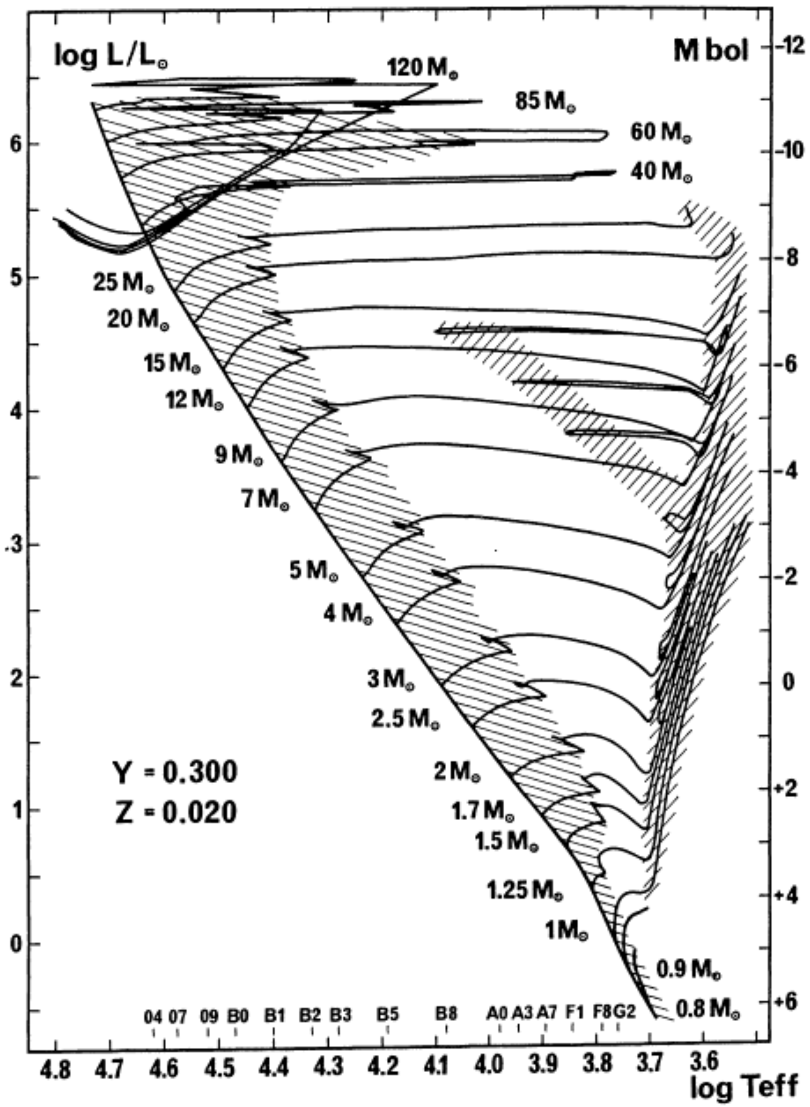
We will derive the 4^{th} of these equations and explore how to solve the equations of stellar structure to construct models.

Learning Outcomes

The student will learn:

- How to derive the 4th equation to describe stellar structure
- Explore ways to solve these equations.
- How to go about constructing models of stellar evolution – how the models can be made to be time variable. You will gain an understanding of what time dependent processes are the most important
- How to come up with the boundary conditions required for the solution of the equations.
- How to consider the effects and influence of convection in stars, when and where it is important, and how it can be included into the structure equations.

Theoretical stellar evolution



With Richard Monier, you will discuss the results of modern stellar evolutionary computations.

The outcome will be this type of theoretical HR-diagram.

At present we are deriving the fundamental physics underlying the calculations - the end point is a diagram like this.

The characteristic timescales

There are 3 characteristic timescales that aid concepts in stellar evolution

The dynamical timescale

Derived in Lecture 2:

$$t_d = \left(\frac{2r^3}{GM}\right)^{\frac{1}{2}}$$

For the Sun t_d~2000s

The thermal timescale

Derived in Lecture 4: time for a star to emit its entire reserve of thermal energy upon contraction provided it maintains constant luminosity (Kelvin-Helmholtz timescale)

For the Sun t_{th}~30 Myrs

$$t_{th} \sim \frac{GM^2}{Lr}$$

The nuclear timescale

Time for star to consume all its available nuclear energy (ε = typical nucleon binding energy/nucleon rest mass energy

For Sun t_{nuc} is larger than age of Universe

$$t_{nuc} \sim \frac{\varepsilon M c^2}{L}$$

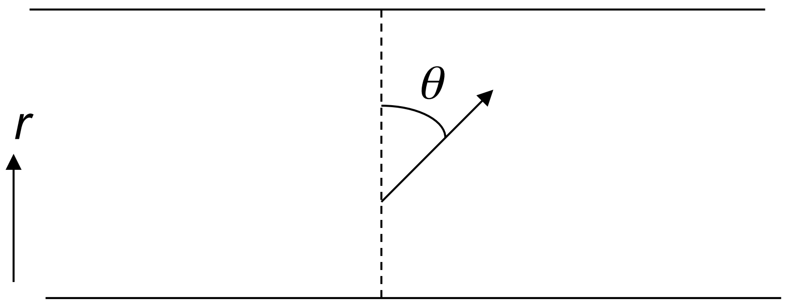
$$\Rightarrow t_d << t_{th} << t_{nuc}$$

The equation of radiative transport

We assume for the moment that the condition for convection is not satisfied, and we will derive an expression relating the change in temperature with radius in a star assuming all energy is transported by radiation. Hence we ignore the effects of convection and conduction.

We will make use of your knowledge of Marianne Faurobert, which covered stellar atmospheres and radiative transport.

Recall the equation of radiative transport in a plane parallel geometry i.e. the gas conditions are a function of only one coordinate, in this case *r*



$$\cos\theta = \mu$$

$$dx = \frac{dr}{\mu}$$

$$\text{or} \quad \frac{dI_\upsilon}{dx} = \mu \frac{dI_\upsilon}{dr}$$

$$\Rightarrow \mu \frac{dI_\upsilon}{dr} = \kappa_\upsilon \rho (I_\upsilon + \frac{j_\upsilon}{\kappa_\upsilon})$$

The equation of radiative transport

See handout for derivation of equation:

$$\frac{dT}{dr} = \frac{3\rho\kappa_R}{64\pi r^2 \sigma T^3} L(r)$$

Solving the equations of stellar structure

Hence we now have four differential equations, which govern the structure of stars (note – in the absence of convection).

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$

$$\frac{dT(r)}{dr} = -\frac{3\rho(r)\kappa_R(r)}{64\pi r^2 \sigma T(r)^3} L(r)$$

Where

r = radius

P = pressure at *r*

M = mass of material within *r*

ρ = density at *r*

L = luminosity at *r* (rate of energy flow across sphere of radius *r*)

T = temperature at *r*

κ_R = Rosseland mean opacity at *r*

ε = energy release per unit mass per unit time

We will consider the quantities:

P = P (ρ, T, chemical composition) **The equation of state**

$\kappa_R = \kappa_R$*(ρ, T, chemical composition)*

$\varepsilon = \varepsilon$ *(ρ, T, chemical composition)*

Boundary conditions

Two of the boundary conditions are fairly obvious, at the centre of the star
M=0, L=0 at r=0

At the surface of the star its not so clear, but we use approximations to allow solution. There is no sharp edge to the star, but for the the Sun ρ(surface)~10^{-7} g cm^{-3}. Much smaller than mean density ρ(mean)~1.4 g cm^{-3} (which we derived). We know the surface temperature (T_{eff}=5780K) is much smaller than its minimum mean temperature (2×10^6 K).

Thus we make two approximations for the surface boundary conditions:
$\rho = T = 0$ at $r=r_s$
i.e. that the star does have a sharp boundary with the surrounding vacuum

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of *M* rather than *r.*

If we divide the other three equations by the equation of mass conservation, and invert the latter:

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{dT}{dM} = -\frac{3\kappa_R L}{64\pi^2 r^4 acT^3}$$

With boundary conditions:

r=0, L=0 at M=0

$\rho=0$, $T=0$ at $M=M_s$

We specify M_s and the chemical composition and now have a well defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

Stellar evolution

We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition. However what do these equations not provide us with ?

In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time, we can ignore temporal changes (e.g. pulsations)

$$t_d = \left(\frac{2r^3}{GM} \right)^{\frac{1}{2}}$$

And for the Sun for example, this is t_d~2000s, hence this is certainly true

And we have also made the assumption that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}

Stellar evolution

If there are no bulk motions in the interior of the star, then any changes of chemical composition are localised in the element of material in which the nuclear reactions occurred. So star would have a chemical composition which is a function of mass *M.*

In the case of no bulk motions – the set of equations we derived must be supplemented by equations describing the rate of change of *abundances* of the different chemical elements. Let $C_{X,Y,Z}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (X), helium, (Y) and metals (Z) [e.g. for solar system X=0.7, Y=0.28, Z=0.02]

$$\frac{\partial (C_{X,Y,Z})_M}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

Now lets consider how we could evolve a model

$$(C_{X,Y,Z})_{M,t_0+\delta t} = (C_{X,Y,Z})_{M,t_0} + \frac{\partial (C_{X,Y,Z})_M}{\partial t}$$

Convection

When the radial flux of energy is carried by radiation, we derived an expression for the temperature gradient:

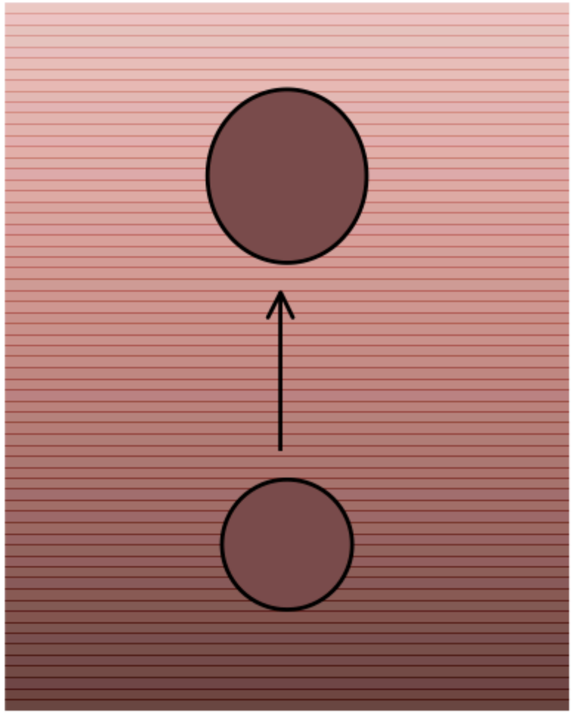
$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

Large luminosity and / or a large opacity κ implies a large (negative) value of dT / dr.

For an ideal gas, the energy density (energy per unit volume) is given by: $\frac{3}{2}nkT$ …with n the number density of particles.

Hot gas near the center of the star has higher energy density than cooler gas above - if we could `swap' the gas over we could transport energy outward… especially if dT / dr is large.

Schwarzschild criteria for convective instability



Imagine displacing a small mass element vertically upward by a distance dr. Assume that **no heat** is exchanged with the surrounding, i.e. the process is **adiabatic**:

- Element expands to stay in pressure balance with the new environment
- New density will *not* generally equal the ambient density at the new location

If this mechanical energy transport is more efficient than the radiative case, the medium will be **convectively unstable**

Stability condition is:

Temperature gradient in the star

$$\left|\left(\frac{dT}{dr}\right)_{star}\right| < \left|\left(\frac{dT}{dr}\right)_{adiabatic}\right|$$

Temperature gradient when an element is moved adiabatically

The important physical point is:

Too steep a temperature gradient leads to the onset of convection in stars

Since a steep gradient is caused by a large luminosity, can convert this into an expression for the **maximum** luminosity that can be transported radiatively:

$$L_{\max} \propto \frac{1}{\kappa}\left(1 - \frac{1}{\gamma}\right)$$

omitting lots of factors but keeping the important dependencies on opacity and adiabatic exponent

Larger luminosities lead to convection.

Which stars are convectively unstable?

Low mass stars

$$L_{\max} \propto \frac{1}{\kappa}\left(1-\frac{1}{\gamma}\right)$$

Near the surface, opacity is large (atomic processes) and $\gamma < 5/3$ due to ionization. Leads to **surface convection zones**.

High mass stars

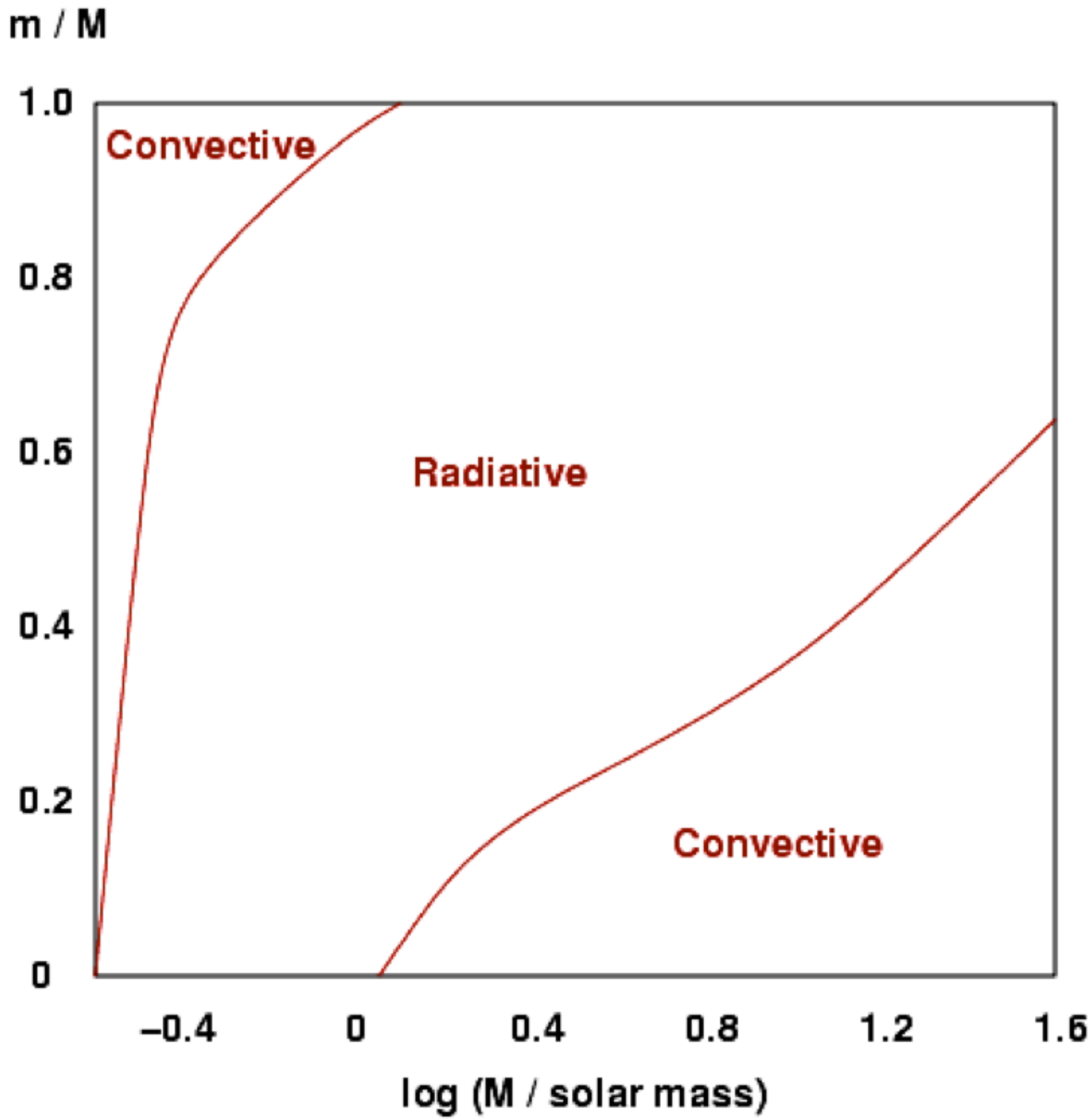
Luminosity of stars increases very rapidly with increasing stellar mass: $L \sim M^4$ for stars of around a Solar mass.

All this energy is generated very close to the core of the star. Can exceed the critical value - **core convection**.

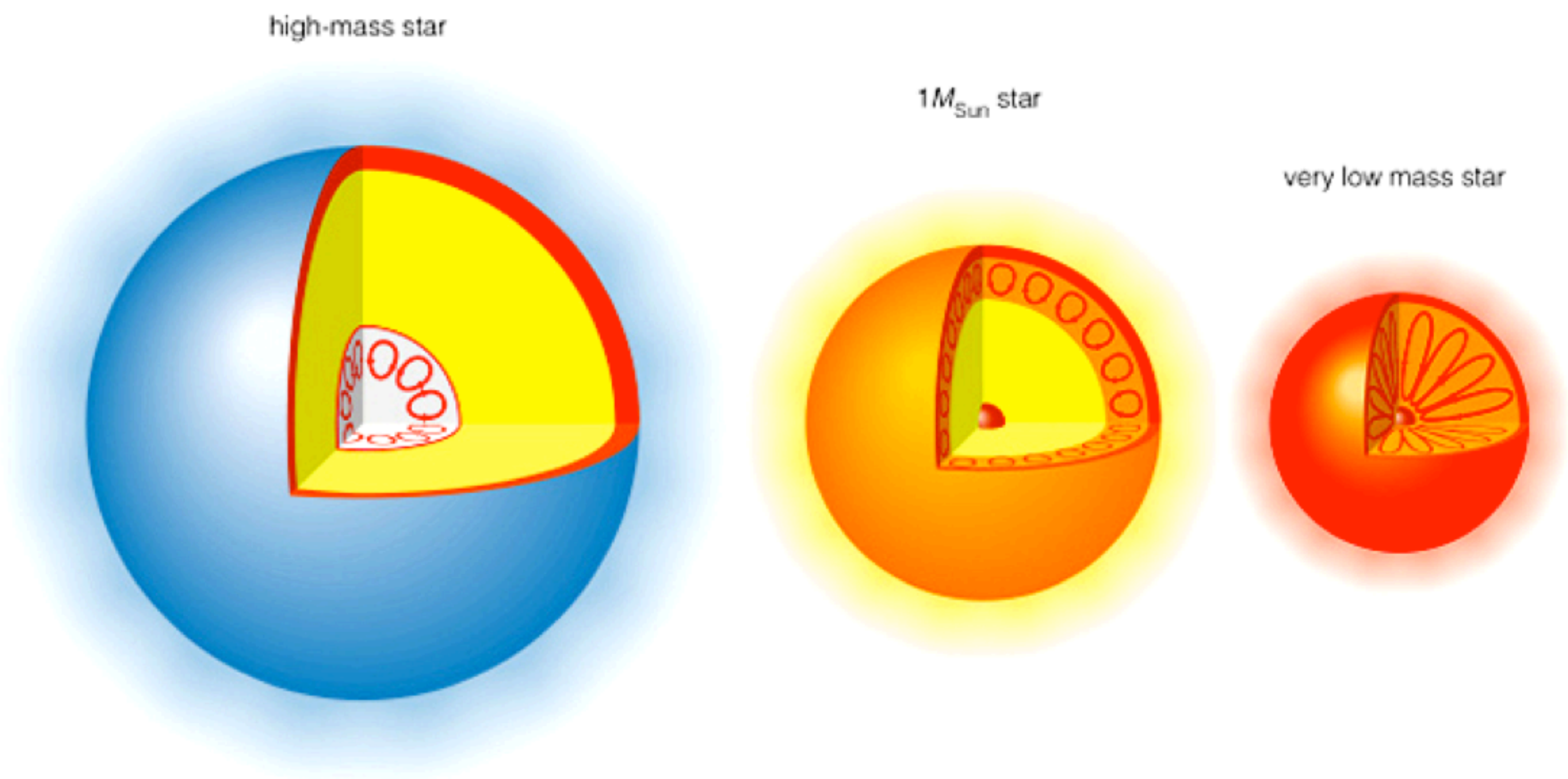
Pre-main-sequence stars

Fully convective due to the large dissipation of gravitational potential energy as they contract.

Regions of convection in main sequence stars



Differences in Stellar Structures



Thus IN A CONVECTIVE REGION we must solve the four differential equations, together with equations for ε and *P*

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

The eqn for luminosity due to radiative transport is still true:

$$L_{rad} = \frac{64\pi^2 r^4 acT^3}{3\kappa_R}\frac{dT}{dM}$$

And once the other equations have been solved, L_{rad} can be calculated. This can be compared with *L* (from $dL/dM = \varepsilon$) and the difference gives the value of luminosity due to convective transport $L_{conv} = L - L_{rad}$

In solving the equations of stellar structure the eqns appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.

Conclusions and summary

We have derived the 4th equation to describe stellar structure, and explored the ways to solve these equations.

As they are not time dependent, we must iterate with the calculation of changing chemical composition to determine short steps in the lifetime of stars. The crucial changing parameter is the H/He content of the stellar core (and afterwards, He burning will become important – to be explored in Richard Monier’s lectures).

We have discussed the boundary conditions applicable to the solution of the equations and made approximations, that do work with real models.

We have explored the influence of convection on energy transport within stars and have shown that it must be considered, but only in areas where the temperature gradient approaches the adiabatic value. In other areas, the energy can be transported by radiation alone and convection is not required.